Game Transformations That Preserve Nash Equilibria or Best Response Sets

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ABSTRACT

In this paper, we investigate under which conditions normal-form games are (guaranteed) to be strategically equivalent. First, we show for N-player games ($N \geq 3$) that

(A) it is NP-hard to decide whether a strategy constitutes a best response to some strategy profile of the opponents, and that
(B) it is co-NP-hard to decide whether two games have the same best response sets.

Combining that with known results from the literature, we move our attention to equivalence-preserving game transformations.

It is a widely used fact that a positive affine (linear) transformation of the utility payoffs neither changes the best response sets nor the Nash equilibrium set. We investigate which other game transformations also possess either of the two properties when being applied to an arbitrary N-player game ($N \geq 2$):

(i) The Nash equilibrium set stays the same.
(ii) The best response sets stay the same.

For game transformations that operate player-wise and strategy-wise, we prove that (i) implies (ii) and that transformations with property (ii) must be positive affine. The resulting equivalence chain highlights the special status of positive affine transformations among all the transformation procedures that preserve key game-theoretic characteristics.

KEYWORDS

Strategic Equivalence, Game Transformation, Nash Equilibrium, Best Responses, Positive Affine Linear Transformation

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1 INTRODUCTION

When faced with a strategic interaction with other agents, it is helpful to detect when the current situation can be treated in the same way as another strategic game that has already been dealt with in in the past. Du [13] has shown that this is generally a computationally hard task for the case of Nash equilibria. As we will show, this task is also computationally hard in the case of best responses.

Therefore, one may instead take an alternative approach for the currently encountered strategic interaction and generate a space of many other situations that share key game-theoretic characteristics, with the goal to find an instance of that space that can be analyzed and solved efficiently. More concretely, a classic tool that emerged since the beginnings of game theory has been to transform a given game into other strategically equivalent games that are easier to analyze [39]. Positive affine (linear) transformations (PATs) have been particularly useful in that regard [3, 5, 24]. To illustrate PATs, consider any 2-player normal-form game in which the players' utilities are measured in dollars. Then, the best-response strategies of player 1 do not change if her utility payoffs are multiplied by a factor of 5. Moreover, they also do not change if 10 dollars are added to all outcomes that involve player 2 playing his, say, third strategy. More generally, PATs have the power to rescale the utility payoffs of each player and to add constant terms to the utility payoffs of a player i for each strategy choice $\mathbf{k}_{-i}$ of her opponents.

Through leveraging PATs, previous work significantly extended the applicability of efficient Nash equilibrium solvers [2, 4, 11, 38] to classes beyond those of zero-sum and rank-1 games[1] [20, 22, 30]. The key to the success of these extensions was the well-known property of PATs that they do not change the Nash equilibrium set and best response sets when being applied to an arbitrary game.

In this paper, we address the question of whether there are other (efficiently computable) game transformations with that same property.

2 OVERVIEW

Sections 3 and 5 provide some background on game-theoretic concepts that are relevant to understanding and deriving our main results. In Section 4, we develop computational hardness results for deciding whether a strategy in a game ever constitutes a best response and for deciding whether two games have the same best response sets. We believe these results are of independent interest. However, they are also important for Section 6 in which we discuss why we will henceforth restrict our attention to game transformations that transform utilities player-wise and strategy-wise (called *separability*). In Section 7 we proceed to characterize all separable game transformations that preserve the Nash equilibrium set when being applied to an arbitrary N-player game. Last but not least, Section 8 puts our results into context with further related work.

To illustrate the insights of Section 7 on an example, consider H_{Ex} that takes any 2-player 2×2 normal-form game with payoff matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and transforms it into the game $H_{\text{Ex}}(A, B) := (A', B')$ that is defined as

$$A' = \begin{pmatrix} -2 \cdot a_{11} + 10 & a_{12}^5 \\ e^{a_{21}} & 0 \end{pmatrix}$$

[1] A 2-player game, represented by its payoff matrices $A, B \in \mathbb{R}^{m \times n}$, is said to have rank-1 if $\text{rank}(A + B) = 1$.

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and

$$B' = \begin{pmatrix} |b_{11}| & \mathrm{sign}(b_{12}) \\ \sqrt{|b_{22}|} & \arctan(b_{21}) \end{pmatrix} .$$

As one can see with the sign function in B', it is noteworthy to highlight that our notion of a game transformation allows for non-continuous functions. With Theorem 2, we will show that there must exist 2×2 games $(\bar{A}, \bar{B})$ for which H_{Ex} does not preserve their Nash equilibrium set or - respectively - their best response sets. More generally, we derive that *universally* preserving the Nash equilibrium set implies that the best response sets always have to be preserved as well; and that the latter property is only satisfied by game transformations H with the very restricted structure of a PAT. In the example of H_{Ex}, each transformation map within it single-handedly already violates a PAT structure.

All proofs for statements in this paper can be found in the appendix.

3 NORMAL-FORM GAMES

Notation-wise, we denote $[n] := \{1, \dots, n\}$ for any $n \in \mathbb{N}$.
A normal-form multiplayer game G specifies

(a) the number of players $N \in \mathbb{N}, N \geq 2$,
(b) a set of pure strategies $S^i = [m_i]$ for each player i where $m_i \in \mathbb{N}$, $m_i \geq 2$, and
(c) the utility payoffs for each player i given as a function $u_i : S^1 \times \ldots \times S^N \longrightarrow \mathbb{R}$.

Denote the set of strategy profiles in G as $S := S^1 \times \ldots \times S^N$. Throughout this paper, all considered multiplayer games shall have the same number of players N and the same set of strategy profiles S. Hence, any game G will be determined by its utility functions $\{u_i\}_{i \in [N]}$. The players choose their strategies simultaneously and they cannot communicate with each other. A utility function u_i can be summarized by its pure strategy outcomes for player i, captured as an N-dimensional tensor or array $\{u_i(\mathbf{k})\}_{\mathbf{k} \in S}$.

As usual, we allow the players to randomize over their pure strategies, called mixed strategies. Then, player i's strategy space extends to the set of probability distributions

$$\Delta(S^i) := \left\{ s^i = (s^i_k)_k \in \mathbb{R}^{m_i} \,\middle|\, s^i_k \geq 0 \;\forall k \in [m_i] \text{ and } \sum_{k \in [m_i]} s^i_k = 1 \right\}$$

over S^i. A tuple

$$\mathbf{s} = (s^1, \dots, s^N) \in \Delta(S^1) \times \ldots \Delta(S^N) =: \Delta(S)$$

is called a mixed strategy profile[2] in G. The utility payoff of player i under profile $\mathbf{s}$ is defined as the player's utility payoff in expectation

$$u_i(\mathbf{s}) := \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k}) .$$

The goal of each player is to maximize her utility.

We will abbreviate with S^{-i} the set that consists of all possible pure strategy choices $\mathbf{k}_{-i} = (k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_N)$ of the opponent players (resp. $\Delta(S^{-i})$ for the set of mixed strategy choices $\mathbf{s}^{-i} = (s^1, \dots, s^{i-1}, s^{i+1}, \dots, s^N)$). We will also use $u_i(k_i, \mathbf{k}_{-i})$ instead of $u_i(\mathbf{k})$ to stress how player i can only influence her own strategy when it comes to her payoff (resp. $u_i(s^i, \mathbf{s}^{-i})$ instead of $u_i(\mathbf{s})$).

[2]Not to be confused with a correlated strategy: In our notation, $\Delta(S)$ itself is not a simplex of high dimension but only the product of N lower-dimensional simplices.

Definition 3.1. The best response set of player i to the opponents' strategy choices $\mathbf{s}^{-i}$ is defined as

$$\mathrm{BR}_{u_i}(\mathbf{s}^{-i}) := \operatorname*{argmax}_{t^i \in \Delta(S^i)} \left\{ u_i(t^i, \mathbf{s}^{-i}) \right\} .$$

Best response strategies capture the idea of optimal play against the other player's strategy choices. The most popular equilibrium concept in non-cooperative games is based on best responses.

Definition 3.2. A strategy profile $\mathbf{s} \in \Delta(S)$ to a game $G = \{u_i\}_{i \in [N]}$ is called a Nash equilibrium if for all player $i \in [N]$ we have $s^i \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i})$.

By Nash [31], any multiplayer game G admits at least one Nash equilibrium.

4 DECISION PROBLEMS ABOUT BEST RESPONSES

In this section we show that two decision problems about best responses are hard for N-player games, when $N \geq 3$. To our knowledge, these results are novel.

For computational problems involving N-player games G with strategy sets $(S^i)_{i \in [N]}$ and utility functions $(u_i)_{i \in [N]}$, we are interested in their computational complexities in terms of $\sum_i |S^i|$ and the binary encoding of all utility payoffs $\left(u_i(\mathbf{s})\right)_{\mathbf{s} \in S, i \in [N]}$. For that, we require that utility payoffs take on rational values only.

First, we consider the problem of deciding whether a mixed strategy of a player is ever a best response to some mixed strategy profile of the opponent players. In its computationally easiest form, we may formulate it as the following.

Definition 4.1 (CheckIfEverBR)**.** Given a 3-player normal-form game, does there exist mixed strategies $\mathbf{r} \in \Delta(S^2)$ of PL2 and $\mathbf{s} \in \Delta(S^3)$ of PL3 such that pure strategy 1 of PL1 is a best response to $(\mathbf{r}, \mathbf{s})$?

This is different from determining the best responses of a player to a given strategy profile of the opponents, a task that can be solved in polynomial time. Our problem is related to *rationalizable* strategies [7, 32] - a concept that is based on the idea that a rational player can and should eliminate any strategy that is not a best response to some belief over what her opponents may play.

Proposition 4.2. CheckIfEverBR *is* **NP**-*hard.*

The analogous formulation of CheckIfEverBR for the case of 2-player games can be efficiently decided by solving a system of linear (in-)equalities. We can recover polynomial-time solvability for many-player games if we allow the opponents to play in a coordinated fashion (cf. correlated strategies). On a related note, Pearce [32][Lemma 3] shows that a strategy s^* is a best-response to some correlated strategy of the opponents if and only if s^* is not a strictly dominated strategy.

We prove Proposition 4.2 by a reduction from the Balanced Complete Bipartite Subgraph problem. This decision problem asks whether a given weighted bipartite graph $G = (V \cup W, E)$ has subsets $V^* \subseteq V$ and $W^* \subseteq W$ of given size $K \in \mathbb{N}$ that are fully connected, that is, $(v, w) \in E$ for all $v \in V^*$, $w \in W^*$. This problem is known to be **NP**-complete [17][GT24].

Proof sketch of Proposition 4.2. Given an instance $G = (V \cup W, E)$ and K of the Balanced Complete Bipartite Subgraph problem, construct a three player game where PL2 has strategy set V and PL3 has strategy set W. PL1 will have the following strategies: Strategy "1" which will be the subject of interest in CheckIfEverBR, one strategy for each node in G, and one strategy for each edge $(v, w) \in V \times W$ that is not present in G. The utility payoffs of PL1 will be carefully constructed such that strategy 1 is a best response to mixed strategies $(\mathbf{r}, \mathbf{s})$ of PL2 and PL3 if and only if the support of $\mathbf{r}$ and $\mathbf{s}$ form subsets V^* and W^* that make a balanced complete bipartite subgraph of G. To that end, we make strategy v (resp. w) of PL1 very attractive for PL1 in the case that PL2 (resp. PL3) plays their corresponding strategy v (resp. w) with too much probability. Moreover, we make a strategy $(v, w) \notin E$ of PL1 very attractive for PL1 in the case that PL2 and PL3 both play their corresponding strategies v and w with any significant probability at all. Intuitively, these two conditions accomplish that in any potential certificate $(\mathbf{r}, \mathbf{s})$, PL2 and PL3 will mix over at least K strategies and, moreover, they will only put non-negligible weight on strategies v and w if $(v, w) \in E$. □

Based on the hardness of CheckIfEverBR, we can prove **co-NP**-hardness of deciding best response equivalence.

Definition 4.3 (CheckIfSameBRs)**.** Given two 3-player normal-form games with strategy set $S^1 \times S^2 \times S^3$, do they have the same best response sets?

Theorem 1. CheckIfSameBRs *is* **co-NP**-*hard.*

Proof sketch. Given a game instance G of CheckIfSameBRs, construct another game G' by changing the utility that PL1 receives from playing strategy 1 to something worse than the lowest payoff present in G. If a best response set changed from G to G', then it must also be the case that strategy 1 for PL1 was added or removed from that best response set. The former cannot happen because strategy 1 is strictly dominated for PL1 in G' which prevents it from ever being a best response. Thus, G and G' will have the same best response sets if and only if strategy 1 is never a best response strategy in G. □

Together with prior work found in the literature, Theorem 1 will guide us in the next sections when it comes to the types of game transformations that we may consider for preserving key game-theoretic characteristics. We believe, however, that Proposition 4.2 and Theorem 1 are of independent interest for algorithmic game theory and AI research.

5 PRELIMINARIES ON GAME TRANSFORMATIONS

We introduce PATs and the more general class of separable game transformations.

5.1 Positive Affine Transformations

The following lemma (or restricted versions of it) is a well-known result for 2-player games[3]. Here, the notation $\mathbf{1}_n \in \mathbb{R}^n$ stands for the vector with all ones as its entries.

Lemma 5.1. *Let (A, B) be an $m_1 \times m_2$ bimatrix game and take arbitrary scalars $\alpha_1, \alpha_2 > 0$ and vectors $c^1 \in \mathbb{R}^{m_2}$, $c^2 \in \mathbb{R}^{m_1}$. Define*

$$A' = \alpha_1 A + \mathbf{1}_{m_1} (c^1)^T \quad \text{and} \quad B' = \alpha_2 B + c^2 \mathbf{1}_{m_2}^T \ .$$

Then (A', B') has the same best response sets as (A, B). Consequently, both games have the same Nash equilibrium set.

The game transformations in Lemma 5.1 are called (2-player) positive affine transformations (PATs). An explicit example of a 2-player PAT is one that transforms a 2×2 game (A, B) into

$$A' = \begin{pmatrix} 2a_{11} + 10 & 2a_{12} - 5 \\ 2a_{21} + 10 & 2a_{22} - 5 \end{pmatrix}$$

and

$$B' = \begin{pmatrix} \frac{1}{2} b_{11} & \frac{1}{2} b_{12} \\ \frac{1}{2} b_{11} - \sqrt{3} & \frac{1}{2} b_{21} - \sqrt{3} \end{pmatrix} .$$

The intuition behind Lemma 5.1 is as follows: PL1 wants to maximize her utility given the strategy that PL2 chose. A positive rescaling of u_1 will change the utility payoffs but will not change the utility-maximizing strategies. The same holds true if we add utility payoffs to u_1 that are only dependent on the strategy choice of her opponent PL2, because that would make a constant shift in terms of the decision variables of PL1.

Let us generalize PATs to multiplayer games.

Definition 5.2. A positive affine transformation (PAT) specifies for each player i a scaling parameter $\alpha^i \in \mathbb{R}$, $\alpha^i > 0$, and translation constants $C^i := (c^i_{\mathbf{k}_{-i}})_{\mathbf{k}_{-i} \in S^{-i}}$ for each choice of pure strategies from the opponents. The PAT $H_{\text{PAT}} = \left\{\alpha^i, C^i\right\}_{i \in [N]}$ then takes any game $G = \{u_i\}_{i \in [N]}$ as an input and returns the transformed game $H_{\text{PAT}}(G) = \{u'_i\}_{i \in [N]}$ with utility functions

$$\begin{aligned} u'_i : S &\longrightarrow \mathbb{R} \\ \mathbf{k} &\longmapsto \alpha_i \cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}} \ . \end{aligned} \tag{1}$$

Multiplayer PATs also preserve the best response sets and Nash equilibrium set, which we prove in the appendix for completeness.

Lemma 5.3. *Take a PAT $H_{\text{PAT}} = \left\{\alpha^i, C^i\right\}_{i \in [N]}$ and any game $G = \{u_i\}_{i \in [N]}$. Then, the transformed game $H_{\text{PAT}}(G) = \{u'_i\}_{i \in [N]}$ has the same best response sets as the original game G. Consequently, $H_{\text{PAT}}(G)$ also has the same Nash equilibrium set as G.*

PATs have found much success as a tool for simplifying a given game precisely because of this property. We want to investigate which other game transformations also preserve the best response sets or the Nash equilibrium set. If we found more of these transformations, we could use them to, e.g., further increase the class of efficiently solvable games.

[3]See Heyman and Gupta [20, Lemma 2.1], Maschler et al. [26, Theorem 5.35], Harsanyi and Selten [19, Chapter 3] or Başar and Olsder [6, Proposition 3.1].

5.2 Separable Game Transformations

In this paper, we will focus on the following space of game transformations. We discuss in Section 6 why this forms a maximally large search space within which we may still reasonably hope to find game transformation that are equivalence-preserving and efficiently computable.

Definition 5.4. A separable game transformation $H = \{H^i\}_{i \in [N]}$ specifies for each player i a collection of functions

$$H^i := \left\{ h^i_{\mathbf{k}} : \mathbb{R} \longrightarrow \mathbb{R} \right\}_{\mathbf{k} \in S},$$

indexed by the different pure strategy profiles $\mathbf{k}$.

The transformation H can then be applied to any N-player game $G = \{u_i\}_{i \in [N]}$ with strategy set S to construct the transformed game $H(G) = \{H^i(u_i)\}_{i \in [N]}$ where

$$H^i(u_i) : S \to \mathbb{R}, \quad \mathbf{k} \mapsto h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big) . \tag{2}$$

Observe that the utility payoff of player i in the transformed game $H(G)$ from the pure strategy outcome $\mathbf{k}$ is only a function of the utility payoff from that same player in that same pure strategy outcome of the original game G.

We extend the utility functions $H^i(u_i)$ to mixed strategy profiles $\mathbf{s} \in \Delta(S)$ as usual through

$$H^i(u_i)(\mathbf{s}) := \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^N_{k_N} \cdot h^i_{\mathbf{k}}\big(u_i(\mathbf{k})\big) .$$

To simplify future notation, we will often use $h^i_{k_i, \mathbf{k}_{-i}}$ to refer to $h^i_{\mathbf{k}}$.

Remark 5.5. A multiplayer positive affine transformation $H_{\text{PAT}} = \left\{\alpha^i, C^i\right\}_{i \in [N]}$ makes a separable game transformation $H = \{H^i\}_{i \in [N]}$ by setting

$$\begin{aligned} h^i_{\mathbf{k}} :\ & \mathbb{R} \to \mathbb{R} \\ & z \mapsto \alpha^i \cdot z + c^i_{\mathbf{k}_{-i}} . \end{aligned}$$

In the following Definitions 5.6 and 5.7, we define the universally preserving characteristics that we are interested in.

Definition 5.6. Let $H = \{H^i\}_{i \in [N]}$ be a separable game transformation. Then we say that H universally preserves Nash equilibrium sets if for all games $G = \{u_i\}_{i \in [N]}$ the transformed game $H(G) = \{H^i(u_i)\}_{i \in [N]}$ has the same Nash equilibrium set as G.

Definition 5.7. Let map H^i come from a separable game transformation H. Then we say that H^i universally preserves best responses if for all utility functions $u_i : S \longrightarrow \mathbb{R}$ and for all opponents' strategy choices $\mathbf{s}^{-i} \in \Delta(S^{-i})$:

$$\begin{aligned} \text{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) &= \operatorname*{argmax}_{t^i \in \Delta(S^i)} \left\{ H^i(u_i)(t^i, \mathbf{s}^{-i}) \right\} \\ &= \operatorname*{argmax}_{t^i \in \Delta(S^i)} \left\{ u_i(t^i, \mathbf{s}^{-i}) \right\} = \text{BR}_{u_i}(\mathbf{s}^{-i}) . \end{aligned}$$

Lemma 5.3 states that the maps H^i of a PAT universally preserve best responses. Note, moreover, that by definition of a Nash equilibrium, a game transformation $H = \{H^i\}_{i \in [N]}$ will universally preserve Nash equilibrium sets if for all player i the map H^i universally preserves best responses. Therefore, being a PAT implies Definition 5.7 implies Definition 5.6. In Section 7 we will show the reverse implication chain for game transformations that are separable.

6 DISCUSSION OF RESTRICTIONS

The space of separable game transformations forms a vast landscape in which we may search for universally preserving transformations. This can be seen from the game transformation example H_{Ex} of Section 2. However, one might still ask why this paper does not expand its attention to non-separable game transformations. We will discuss that in this section.

For example, consider a game transformation that introduces or removes duplicate strategies or dummy players. Note that this would require the transformations to have the power to change the strategy sets and player set. Nonetheless, these specific examples are well-behaved in the sense that they alter the Nash equilibrium set (or best responses) in an easily describable manner. Transformations with this property appear in the literature under the term *Nash homomorphism*, and they have been used for complexity-theoretic studies, e.g., of win-lose games [1] or ranking games [8]. Suffice to say, once we allow for game transformations to arbitrarily change the game structure, i.e. the player set and strategy sets, it is not straightforward to define anymore under what conditions two games of different game structure should be considered "strategically equivalent". This makes such general game transformations prohibitively complex (or impossible) to analyze beyond a case by case basis. Therefore, and in accordance with most of the literature on strategic equivalence between games [13, 23, 29, 30], we restrict our attention to games whose game structures are directly comparable.

Indeed, game transformations that preserve the player set and the strategy sets form an interesting search space because Definitions 5.6 and 5.7 can be directly extended to it and because within that search space, some of our upcoming results will not hold true anymore. Compare the Prisoner's Dilemma with the Quality game, as presented by von Stengel [40]:

$$\begin{pmatrix} 2,2 & 0,3 \\ 3,0 & 1,1 \end{pmatrix} \text{ and } \begin{pmatrix} 2,2 & 0,1 \\ 3,0 & 1,1 \end{pmatrix} .$$

Both games have the same game structure and the same unique Nash equilibrium, namely, where PL1 plays the bottom row and PL2 plays the right column. But the best response of PL2 to PL1 playing the top row is different in the two games. This example illustrates the fact that strictly dominated strategies will never be a best response, and so they will never appear in a Nash equilibrium (nor in a best response set). Therefore, we can think of a game transformation procedure that iteratively detects strictly dominated strategies and sets their payoffs to a large negative number. This transformation universally preserves Nash equilibria, but it does not universally preserve best response sets. Note that this game transformation is not separable because its maps $h^i_{\mathbf{k}}$ now need to take all utility payoffs of the game into consideration, and not only what utility player i receives from strategy profile $\mathbf{k}$.

In a similar fashion, one may think of best-response-preserving transformations that are not PATs. This was studied extensively by Liu [23], who discusses the following example of 3×2 payoff matrices of PL1 in 2-player games:

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 6 \\ 4 & 4 \end{pmatrix} \text{ and } A' = \begin{pmatrix} 6 & 0 \\ 2 & 5 \\ 4 & 4 \end{pmatrix} . \tag{3}$$

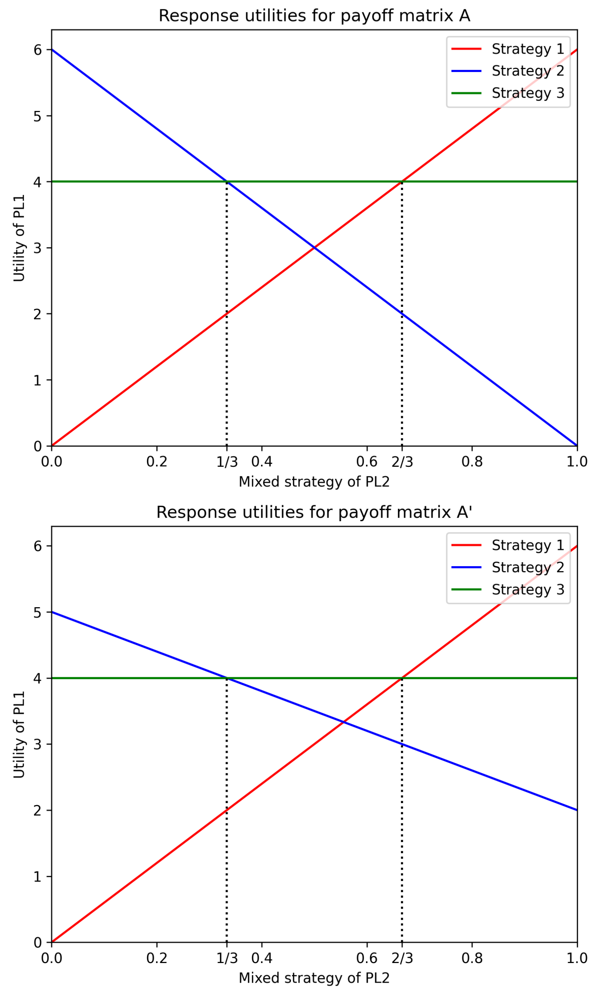

Figure 1: The utility payoffs of each pure strategy $1, 2, 3$ of PL1 in response to the mixed strategy of PL2 that plays 1 with probability x and that plays 2 with probability $1 - x$. Plotted once each for the matrices A and A' from (3). The best response set to a strategy $(x, 1 - x)$ of PL2 will be all convex combinations of pure strategies of PL1 that are maximal at value x in the respective plot.

As visualized by Figure 1, the best responses of PL1 to any mixed strategy of PL2 are the same in A and A'. However, A' cannot be obtained from A through a PAT: If there were such a PAT, then the payoff from profile $(2, 1)$ requires a shift of $c_1^1 = 2$. Hence, the payoff from profile $(1, 1)$ requires a scaling of $\alpha^1 = \frac{2}{3}$. But these components of a positive affine transformation do not work out for the payoff from profile $(3, 1)$, leaving us with a contradiction.

Liu [23] develops a polynomial-time method, called *bi-affine* transformation, that determines whether two 2-player normal-form games have the same best response sets. Their procedure detects which strategies and strategy pairs are essential, and derives that only the essential pairs need to be in a positive affine relationship. Hence, their method includes PATs, but it is also more powerful than that. In their PhD thesis, they extend their ideas to N-player games ($N \geq 3$). But in those games, their method downgrades to a sufficient condition: Two N-player games ($N \geq 3$) may have the same best response sets while not being a *quasi-affine* transformation of each other. Furthermore, their method becomes computationally inefficient. In fact, we have shown in Theorem 1 more generally that determining whether two 3-players games have the same BR sets is **co-NP**-hard.

Liu concludes with an immediate open problem for future work: to characterize games with the same Nash equilibria. To that end, Du [13] proves that it is **NP**-complete to decide whether two 2-player games share a common Nash equilibrium, and that it is **co-NP**-hard to decide whether two 2-player games have the same Nash equilibrium set.

In light of these negative results about characterizing best-response equivalence and Nash equilibrium equivalence in full generality - assuming the well-accepted complexity belief **co-NP** $\neq$ **P** - we restrict our focus to a subclass of equivalence-preserving transformations based on separability. We argue that among naturally defined subclasses, separable game transformations compose the most maximal subclass for which it is still open whether it contains tractable and equivalence-preserving transformations aside from PATs.

7 TRANSFORMATIONS THAT PRESERVE NASH EQUILIBRIUM SETS OR BEST RESPONSE SETS

To our knowledge, the results of this section are all novel unless explicitly stated otherwise. They can be summarized in the following statement.

Theorem 2. *Let $H = \{H^i\}_{i \in [N]}$ be a separable game transformation. Then:*

$$\begin{aligned} & H \text{ universally preserves Nash equilibrium sets} && \text{(i)} \\ \iff & \text{for each player } i, \text{ map } H^i \text{ universally} && \text{(ii)} \\ & \text{preserves best responses} && \\ \iff & H \text{ is a positive affine transformation.} && \text{(iii)} \end{aligned}$$

Lemma 5.3 gives (iii) $\implies$ (i), and so the novel part of Theorem 2 is the implication chain (i) $\implies$ (ii) $\implies$ (iii). The key property that enables us to develop this chain is that we require the separable game transformations $H = \{H^i\}_{i \in [N]}$ to be *universally* applicable, no matter the game $G = \{u_i\}_{i \in [N]}$ we have at hand.

First, let us characterize a special property that a game transformation can satisfy in which the strategy choice of player i does not influence the map that is being used to transform her utilities.

Definition 7.1. Let map H^i come from a separable game transformation H. Then we say that H^i only depends on the strategy choices of the opponents if for all pure strategy choices $\mathbf{k}_{-i} \in S^{-i}$ of the opponents, we have the map identities

$$h^i_{1,\mathbf{k}_{-i}} = \ldots = h^i_{m_i,\mathbf{k}_{-i}} : \mathbb{R} \to \mathbb{R}\,.$$

Next, we can show (i) $\implies$ (ii).

Proposition 7.2. *Let $H = \{H^i\}_{i\in[N]}$ be a separable game transformation that universally preserves Nash equilibrium sets and consider the map H^i of a player i. Then H^i only depends on the strategy choices of the opponents. Furthermore, H^i universally preserves best responses.*

Proof sketch.
First conclusion: Such a universally preserving transformation H should in particular not change the Nash equilibrium set for a trivial game in which all players receive the same constant utility $z \in \mathbb{R}$ from all strategy profiles. In such a game, the whole strategy set S will make the Nash equilibrium set. For that to also be the case in the transformed game, we show for all player i, that the transformations maps $h^i_{1,\mathbf{k}_{-i}}, \dots, h^i_{m_i,\mathbf{k}_{-i}}$ must all evaluate the same on any input value z.
Second conclusion: Let u_i be an arbitrary utility function of player i. Complete u_i to a full game G by setting the utilities of all other players to the constant payoff of 0. This makes any strategy s^j of another player $j \neq i$ always a best response strategy in G. We can then show that this must also hold in the transformed game $H(G)$, using the first conclusion. Therefore, we get the following equivalence chain:

(a) a strategy s^i of player i is a best response to a profile s^{-i} of the opponent players and with respect to u_i if and only if
(b) (s^i, s^{-i}) is a Nash equilibrium of G if and only if
(c) (s^i, s^{-i}) is a Nash equilibrium of $H(G)$ if and only if
(d) s^i a best response to s^{-i} with respect to $H^i(u_i)$.

□

The first conclusion captures the intuition that if the maps $h^i_{\mathbf{k}}$ from H^i would depend on the strategy choice of player i, then in the transformed game $H(G)$, player i may need to adjust her strategy choice to those $h^i_{\mathbf{k}}$ that map payoffs to high values. This would affect the strategic decision making of player i and therefore the Nash equilibrium set overall. Similar reasoning provides us with a related (but independent) result:

Lemma 7.3. *Suppose a map H^i universally preserves best responses. Then H^i only depends on the strategy choices of the opponents.*

Due to Proposition 7.2, we can transition to the analysis of transformation maps H^i that universally preserve best responses. Thus from now on, our results also become relevant to game theory research that focuses on best response sets, such as best response dynamics or fictitious play.

Proposition 7.2 moreover allows us to restrict our analysis to the map H^1 for PL1 w.l.o.g. because any results for H^1 will analogously also hold for maps $H^2, \dots, H^N$. By Lemma 7.3, we can also drop the dependence of H^1 on k_1 and write

$$H^1 := \left\{ h^1_{\mathbf{k}_{-1}} : \mathbb{R} \longrightarrow \mathbb{R} \right\}_{\mathbf{k}_{-1}\in S^{-1}} .$$

For each pure-strategy map $h^1_{\mathbf{k}_{-1}}$ we introduce its *distance distortion* function which takes two utility values and measures their distance after a $h^1_{\mathbf{k}_{-1}}$-transformation:

$$\begin{aligned} \Delta h^1_{\mathbf{k}_{-1}} : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (z, w) &\longmapsto h^1_{\mathbf{k}_{-1}}(z) - h^1_{\mathbf{k}_{-1}}(w) . \end{aligned} \tag{4}$$

The upcoming lemma reveals an important preliminary observation on how the distance distortion functions $\Delta h^1_{\mathbf{k}_{-1}}$ relate to each other. It highlights how the distorted utility distances are connected upon a strategy change of a player $j \neq 1$ from, e.g., some pure strategy $k_j \neq 1$ to their first pure strategy $1 \in [m_j]$. It is again crucial that H^1 preserves best responses *universally* in order to deduce these global properties of and connections between the maps within H^1.

Lemma 7.4. *Suppose transformation map H^1 universally preserves best responses. Take a player $j \in [N]\setminus\{1\}$ and profile $\mathbf{k}_{-1} \in S^{-1}$ with $k_j \neq 1$. Define $\mathbf{k}'_{-1} \in S^{-1}$ to be the same as $\mathbf{k}_{-1}$ except for player j's choice which shall be set to $k'_j = 1$. Then, for all $z, z', w, w' \in \mathbb{R}$:*

$$z - w \geq z' - w' \iff \Delta h^1_{\mathbf{k}_{-1}}(z, w) \geq \Delta h^1_{\mathbf{k}'_{-1}}(z', w') . \tag{5}$$

Proof sketch. Construct a utility function u^1 for each set of values for $j, \mathbf{k}_{-1}, z, z', w$, and w'. Namely, set $u_1(1, \mathbf{k}_{-1}) := z$ and $u_1(1, \mathbf{k}'_{-1}) := w'$, and for all strategies $l \in [m_1]\setminus\{1\}$ set $u_1(l, \mathbf{k}_{-1}) := w$ and $u_1(l, \mathbf{k}'_{-1}) := z'$. Observe that uniformly randomizing over $\mathbf{k}_{-1}$ and $\mathbf{k}'_{-1}$ not only makes a correlated strategy of the opponents, but also a valid mixed strategy profile. Hence, the left hand side of (5) can be reinterpreted as strategy $1 \in [m_1]$ performing better for player 1 than any other of her strategies $l \in [m_1]\setminus\{1\}$ if player j uniformly mixes over strategies k_j and k'_j and if all other players $r \notin \{1, j\}$ play their respective strategy $k_r \in [m_r]$. We then derive equivalence (5) by using that H^1 preserves strategy 1 being such a best response and by using Lemma 7.3. □

Next, observe that by definition, these distance distortion functions are skew-symmetric, that is,

$$\forall\, z, w \in \mathbb{R} : \quad \Delta h^1_{\mathbf{k}_{-1}}(z, w) = -\Delta h^1_{\mathbf{k}_{-1}}(w, z) .$$

With the upcoming lemma, we further tighten the connection between the pure-strategy maps $h^1_{\mathbf{k}_{-1}}$ through their distance distortion functions. Last but not least, we shine some light on how those maps $h^1_{\mathbf{k}_{-1}}$ behave individually in the subsequent lemma.

Lemma 7.5. *Suppose transformation H^1 universally preserves best responses. Then the pure-strategy maps in H^1 equally distort distances:*

$$\forall \mathbf{k}_{-1} \in S^{-1} \,:\, \Delta h^1_{\mathbf{k}_{-1}} = \Delta h^1_{\mathbf{1}_{-1}}$$

where $\mathbf{1}_{-1} := (1, \dots, 1) \in S^{-1}$.

Proof sketch. Make iterative use of Lemma 7.4 for all other players $j \neq 1$, and make use of the skew-symmetry. □

Lemma 7.6. *Suppose transformation H^1 universally preserves best responses. Then we obtain for all $\mathbf{k}_{-1} \in S^{-1}$ that*

(1) map $h^1_{\mathbf{k}_{-1}}$ is strictly increasing, and that
(2) map $h^1_{\mathbf{k}_{-1}}$ distorts distances independently of their reference points:

$$\forall z, z', \lambda \in \mathbb{R} \,:\, \Delta h^1_{\mathbf{k}_{-1}}(z + \lambda, z) = \Delta h^1_{\mathbf{k}_{-1}}(z' + \lambda, z') .$$

Proof sketch. For the first conclusion make use of Lemma 7.4 for values $z' = w'$, and of Lemma 7.5. For the second conclusion, utilize skew-symmetry together with the same two lemmata. □

With Lemmata 7.5 and 7.6, we can finally show that positive affine transformations are the only game transformations that universally preserve best responses. Intuitively speaking, the second conclusion of Lemma 7.6 states that taking a step of length λ in the domain space consistently maps to taking a step of some other length in the range space, independently of the base point z from which we take such a step. This brings us to two known results from the analysis literature. Recall that a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is called linear if there exists some $a \in \mathbb{R}$ such that

$$\forall z \in \mathbb{R} \,:\, h(z) = az \,.$$

A function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be additive if it satisfies

$$\forall x, y \in \mathbb{R} \,:\, h(x + y) = h(x) + h(y) \,.$$

Lemma 7.7 ([12, 36]). *If a map $h : \mathbb{R} \longrightarrow \mathbb{R}$ is monotone and additive, then it is also linear.*

Corollary 7.8. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be monotone and satisfy for all $z, z', \lambda \in \mathbb{R}$:*

$$h(z + \lambda) - h(z) = h(z' + \lambda) - h(z') \,.$$

Then h is affine linear, i.e., there exist some $a, c \in \mathbb{R}$ such that for all

$$\forall z \in \mathbb{R} \,:\, h(z) = az + c \,.$$

This brings us to the completion of this section.

Proof sketch of Theorem 2.
Implication (iii) $\Longrightarrow$ (i) follows from Lemma 5.3, and implication (i) $\Longrightarrow$ (ii) follows from Proposition 7.2. For (ii) $\Longrightarrow$ (iii), recall that by symmetry, our results for H^1 hold analogously for all maps H^i. By Lemmata 7.3 and 7.6, the maps $h^i_{\mathbf{k}} = h^i_{\mathbf{k}_{-i}}$ satisfy the conditions of Corollary 7.8. Thus, there exist parameters $a^i_{\mathbf{k}_{-i}}, c^i_{\mathbf{k}_{-i}} \in \mathbb{R}$ for each $\mathbf{k}_{-i} \in S^{-i}$ such that

$$\forall z \in \mathbb{R} \,:\, h^i_{\mathbf{k}_{-i}}(z) = a^i_{\mathbf{k}_{-i}} \cdot z + c^i_{\mathbf{k}_{-i}} \,.$$

Lemma 7.5 implies $a^i_{\mathbf{k}_{-i}} = a^i_{\mathbf{1}_{-i}}$ for all $\mathbf{k}_{-i} \in S^{-i}$. Therefore, we only have to keep track of one scaling parameter α^i for all the maps within H^i. With the first conclusion of Lemma 7.6, we obtain $\alpha^i > 0$. Putting everything together, we have shown that $H = (H^1, \ldots, H^N)$ makes a positive affine transformation.

□

Theorem 2 gives two novel equivalent characterizations of PATs that highlight their special status among game transformations: PATs are the only types of separable game transformations that always preserve the Nash equilibrium set or, respectively, the best response sets.

One way to circumvent this result is to focus on game transformations that we only care to apply on particular subclasses of N-player games. Preferably, the game properties defining such a subclass would be generic enough to still contain "most" games. On the other hand, one may instead also consider non-separable game transformations as discussed in Section 6.

8 FURTHER RELATED LITERATURE

Strategic Similarity. Much work has gone into identifying when two games can be considered strategically equivalent.

Strategic similarity, for example, is an important aspect of Potential Games (cf. Monderer and Shapley [28]). Morris and Ui [29] noted that a game G is a weighted potential game if and only if it is the PAT transformation of an identical interest game[4]. This characterization has been used to analyze the Nash equilibria and solvers of potential games. The main contribution of Morris and Ui, however, was to characterize when two given games are *best-response equivalent*, *better-response equivalent* or *von Neumann-Morgenstern equivalent*. Two games are best-response equivalent if they have the same best response sets. Better-response equivalency requires that each player's induced preferences over her strategies - given the strategy choices of her opponents - are the same in both games. This equivalency concept has also been characterized by Moulin and Vial [30]. Von Neumann-Morgenstern equivalency requires that the games only differ by a PAT. Unfortunately, we were not able to base the second part of Theorem 2 on the insights from Morris and Ui because their characterization for best-response equivalence only holds for games that satisfy specific properties.

Hammond [18] described that the strategic decision-making in a game in mixed strategies does not depend on the player's numerical utility values, but solely on the preferences that the utility functions induce over the strategies. In the appendix, we give some further background on utility theory in order to put Hammond's work into our context. Using the Expected Utility Theorem - cf., e.g., Mas-Colell et al. [25]) - Hammond deduced that utility functions that induce the same preferences can only differ up to a positive affine transformation. Note, that the property of preserving the player's preferences is, in general, strictly harder to satisfy than preserving best responses (and, hence, Nash equilibria). Thus, our Theorem 2 generalizes their result to the broader question of strategic equivalence.

Moving to more broader related work, Gabarró et al. [15, 16] gave several complexity-theoretic results to the problem of deciding whether two pure strategy games are *isomorphic* w.r.t. a notion of game transformation that can help us understand the symmetries within a game [19, Chapter 3]. McKinsey [27] and Chang and Tijs [10] studied two notions of game equivalency specific to cooperative games.

Game Transformations. There are other lines of related research that work more explicitly with different notions of transforming a game. For example, Pottier and Nessah [33] take interest in game transformations that convert the Berge-Vaisman equilibria of a game to the Nash equilibria of the transformed game. Game transformations that preserve strategic features were also studied by Wu et al. [41] for Bayesian games and by [9, 14, 21, 37] for extensive-form games.

9 CONCLUSION

In this paper, we first gave hardness results about deciding whether a strategy constitutes a best response or whether two games have

[4]Identical interest game: Given an action profile s, each player shall receive the same utility from s.

the same best response sets. Next, we introduced separable game transformations for multiplayer games, and define the properties (i) universally preserving Nash equilibrium sets and (ii) universally preserving best responses. It is well-known that PATs universally preserve Nash equilibrium sets. We showed that separable game transformations which universally preserve Nash equilibrium sets also universally preserve best responses. In the subsequent results, we derived further that if a separable game transformation universally preserves best responses then it is a positive affine transformation.

When faced with a strategic interaction - whether only once or on a regular basis - it can be highly beneficial to consider equivalent variations of it that are easier to analyze. The current literature on game theory and on decision making in AI are lacking methods to detect or generate such strategic equivalent games. Our discussion and results can explain this observation. Simultaneously, we are able to highlight how special PATs are with regard to Nash equilibrium sets and best response sets. Going forward, we hope that our results can serve as guidance to the development of any such detection or generation toolkit.

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A PROOFS OF SECTION 4

To our knowledge, the results and proofs of this section are all novel unless explicitly stated otherwise.

Definition (CheckIfEverBR)**.** Given a 3-player normal-form game, does there exist mixed strategies $\mathbf{r} \in \Delta(S^2)$ of PL2 and $\mathbf{s} \in \Delta(S^3)$ of PL3 such that pure strategy 1 of PL1 is a best response to $(\mathbf{r}, \mathbf{s})$?

In general, for computational problems involving N-player games G with strategy sets $(S^i)_{i \in [N]}$ and utility functions $(u_i)_{i \in [N]}$, we are interested in their computational complexities in terms of $\sum_i |S^i|$ and the binary encoding of all utility payoffs $\left(u_i(\mathbf{s})\right)_{\mathbf{s} \in S, i \in [N]}$. For that, we require that utility payoffs take on rational values only.

We aim to prove Proposition 4.2:

Proposition. CheckIfEverBR *is* **NP**-*hard.*

We achieve this by a reduction from the BiClique problem. Recall that a bipartite graph $G = (V \cup W, E)$ is an undirected graph such that each edge $e \in E$ has one endpoint in V and the other in W.

Definition (BiClique)**.** Given a bipartite graph $G = (V \cup W, E)$ and integer $1 \leq K \leq m + n$, are there subsets $V^* \subseteq V$ and $W^* \subseteq W$ with $|V^*| = K = |W^*|$ and $(v, w) \in E$ for all $v \in V^*, w \in W^*$?

For problems involving bipartite graphs $G = (V \cup W, E)$, we are interested in their computational complexities in terms of $m := |V|$, $n := |W|$ and $l := |E|$.

The complexity of BiClique is known in the literature.

Lemma (Garey and Johnson [17], Problem GT24)**.** BiClique *is* **NP**-*complete.*

Before we get to the proof of Proposition 4.2, let us give a trivial yes instance and a trivial no instance of CheckIfEverBR. This will be used in the proof.

The trivial yes instance shall be $S^1 = \{1\} = S^2 = S^3$ and $u_1(1,1,1) = 0 = u_2(1,1,1) = u_3(1,1,1)$. Then pure strategy 1 of PL1 is a best response to $(1,1)$.

The trivial no instance shall be $S^1 = \{1,2\}$, $S^2 = \{1\} = S^3$, $u_1(1,1,1) = 0 = u_2(\cdot,1,1) = u_3(\cdot,1,1)$ and $u_1(2,1,1) = 1$. Then pure strategy 1 is strictly dominated by 2 and therefore never a best response to a profile of the opponents.

Proof of Proposition 4.2.
Reduction from BiClique. Let $G = (V \cup W, E)$ and $1 \leq K \leq m + n$ be the BiClique instance.

Trivial cases: Let us first remove a couple of edge cases. They are not mutually exclusive, but one can just check these case conditions in the following order until one is satisfied (if at all).

Case 1: If $K \geq \min\{m, n\} + 1$. Then we have a no instance of BiClique. So construct the trivial no instance of CheckIfEverBR.

Case 2: If $K = m$, check in $O(nm)$ time by going through W if there are at least K-many vertices $w \in W$ that satisfy $(v, w) \in E$ for all $v \in V$. If so, then we have a yes instance of BiClique by setting $V^* = V$ and W^* equal to the K-many found w's. So construct the trivial yes instance of CheckIfEverBR. If they do not exist, however, then we have a no instance of BiClique because we couldn't find set W^* of size K that matches the only possibility $V^* = V$. So construct the trivial no instance of CheckIfEverBR.

Case 3: If $K = n$, do the analogous procedure as in Case 2, except with reversed roles for v and w.

Case 4: If $K = m - 1$, check in $O(mnm)$ time if there exists $\bar{v} \in V$ such that are at least K-many vertices $w \in W$ that satisfy $(v, w) \in E$ for all $v \in V \setminus \{\bar{v}\}$. If so, then we have a yes instance of BiClique by setting $V^* = V \setminus \{\bar{v}\}$ and W^* equal to the K-many found w's. So construct the trivial yes instance of CheckIfEverBR. If they do not exist, however, then we have a no instance of BiClique because we couldn't find set W^* of size K that matches the only possibilities $V^* = V \setminus \{\bar{v}\}$ for some $\{\bar{v}\} \in V$. So construct the trivial no instance of CheckIfEverBR.

Case 5: If $K = n - 1$, do the analogous procedure as in Case 4, except with reversed roles for v and w.

Case 6: If neither of the previous case conditions are satisfied. The rest of this proof is considering this case now.

Construction of the corresponding CheckIfEverBR *instance:* Set $S^2 = V$, $S^3 = W$ and $S^1 = \{1\} \cup \{v\}_{v \in V} \cup \{w\}_{w \in W} \cup \{(v, w)\}_{(v,w) \notin E}$. Intuitively, we want to interpret a mixed strategy $\mathbf{r}$ of PL2 as PL2 choosing the support $\text{supp}(\mathbf{r}) := \{v \in V : \mathbf{r}(v) > 0\}$ as the subset V^* of V for the biclique. Analogously, the support of $\mathbf{s}$ of PL3 shall give the subset W^* of W for the biclique. We make a strategy v (resp. w) of PL1 very attractive for PL1 in the case that PL2 (resp. PL3) play their corresponding strategy v (resp. w) with too much probability. This accomplishes that in any potential certificate $(\mathbf{r}, \mathbf{s})$, PL2 and PL3 mix over at least K strategies. We also make a strategy $(v, w) \notin E$ of PL1 very attractive for PL1 in the case that PL2 and PL3 both play their corresponding strategies v and w with any significant probability. This accomplishes that in any potential certificate $(\mathbf{r}, \mathbf{s})$, PL2 and PL3 put non-negligible weight on strategies v and w only if $(v, w) \in E$. Let us proceed with the actual utility payoffs.

Set $u_2(\cdot,\cdot,\cdot) = 0 = u_3(\cdot,\cdot,\cdot)$ because those payoffs are irrelevant. Next, set $u_1(1,\cdot,\cdot) = 1$. Finally, set

$$\forall v, v' \in V : \quad u_1(v, v', \cdot) = \begin{cases} K+1 & \text{if } v = v' \\ 0 & \text{if } v \neq v' \end{cases},$$

$$\forall w, w' \in W : \quad u_1(w, \cdot, w') = \begin{cases} K+1 & \text{if } w = w' \\ 0 & \text{if } w \neq w' \end{cases},$$

and

$$\forall (v, w) \notin E : \quad u_1\Big((v, w), v', w'\Big) =$$
$$\begin{cases} (m-K)(n-K)(K+1)^2 & \text{if } (v, w) = (v', w') \\ 0 & \text{if } (v, w) \neq (v', w') \end{cases}.$$

Note that by assumption of not being in Cases 1, 2, and 3, we have $(m - K)(n - K)(K + 1)^2 > 0$.

Analysis of the corresponding CheckIfEverBR *instance:* First, we observe that for any mixed strategies $\mathbf{r} \in \Delta(S^2)$ of PL2 and $\mathbf{s} \in \Delta(S^3)$ of PL3, we have:

$$\begin{aligned} \forall v \in V : u_1(v, \mathbf{r}, \mathbf{s}) &= \sum_{v' \in V, w' \in W} \mathbf{r}(v')\mathbf{s}(w')u_1(v, v', w') \\ &= \sum_{w' \in W} \mathbf{r}(v)\mathbf{s}(w')(K+1) \\ &= \mathbf{r}(v)(K+1), \end{aligned}$$

and, analogously,

$$\forall w \in W \;:\; u_1(w, \mathbf{r}, \mathbf{s}) = \mathbf{s}(w)(K+1)\,,$$

and

$$\begin{aligned}&\forall (v,w) \notin E \;:\\ &u_1\Big((v,w), \mathbf{r}, \mathbf{s}\Big) = \sum_{v' \in V, w'\; inW} \mathbf{r}(v')\mathbf{s}(w')u_1\Big((v,w), v', w'\Big)\\ &= \mathbf{r}(v)\mathbf{s}(w)(m-K)(n-K)(K+1)^2\,.\end{aligned}$$

Therefore, pure strategy 1 is a best response to $(\mathbf{r}, \mathbf{s})$ if and only if

$$\begin{aligned}\forall v \in V \;:\; \mathbf{r}(v) &= \frac{1}{K+1}u_1(v, \mathbf{r}, \mathbf{s})\\ &\leq \frac{1}{K+1}u_1(1, \mathbf{r}, \mathbf{s}) = \frac{1}{K+1}\,,\end{aligned} \tag{6}$$

$$\begin{aligned}\forall w \in W \;:\; \mathbf{s}(w) &= \frac{1}{K+1}u_1(w, \mathbf{r}, \mathbf{s})\\ &\leq \frac{1}{K+1}u_1(1, \mathbf{r}, \mathbf{s}) = \frac{1}{K+1}\,,\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\forall (v,w) \notin E \;:\\ &\mathbf{r}(v)(m-K)(K+1) \cdot \mathbf{s}(w)(n-K)(K+1)\\ &= u_1\Big((v,w), \mathbf{r}, \mathbf{s}\Big) \leq u_1(1, \mathbf{r}, \mathbf{s}) = 1\,.\end{aligned} \tag{8}$$

Equivalence of BiClique *and its corresponding* CheckIfEverBR *instance:* Suppose the BiClique instance be yes instance that falls into Case 6. Let furthermore V^* and W^* be a biclique certificate. Then, in the corresponding CheckIfEverBR instance, choose the following strategies $(\mathbf{r}, \mathbf{s})$ for PL2 and PL3: for $v \in V$ set

$$\mathbf{r}(v) = \begin{cases} \frac{1}{K+1} & \text{if } v \in V^* \\ \frac{1}{m-K}\frac{1}{K+1} & \text{if } v \notin V^* \end{cases},$$

and for $w \in W$ set

$$\mathbf{s}(w) = \begin{cases} \frac{1}{K+1} & \text{if } w \in W^* \\ \frac{1}{n-K}\frac{1}{K+1} & \text{if } w \notin W^* \end{cases}.$$

Vectors $\mathbf{r}$ and $\mathbf{s}$ form well-defined mixed strategies because we are not in Cases 1, 2, and 3, and because

$$\begin{aligned}\sum_{v \in V} \mathbf{r}(v) &= \sum_{v \in V^*} \mathbf{r}(v) + \sum_{v \notin V^*} \mathbf{r}(v)\\ &= \sum_{v \in V^*} \frac{1}{K+1} + \sum_{v \notin V^*} \frac{1}{m-K}\frac{1}{K+1}\\ &= K\frac{1}{K+1} + (m-K)\frac{1}{m-K}\frac{1}{K+1}\\ &= 1\,,\end{aligned}$$

and analogously $\sum_{w \in W} \mathbf{s}(w) = 1$. Moreover, conditions (6), (7), and (8) are all satisfied. Hence, pure strategy 1 of PL1 is a best response to $(\mathbf{r}, \mathbf{s})$, and, therefore, the corresponding CheckIfEverBR instance a yes instance as well.

Now suppose the BiClique instance falls into Case 6 and the corresponding CheckIfEverBR instanc is a yes instance. Let furthermore $(\mathbf{r}, \mathbf{s})$ be the strategy certificate of PL2 and PL3 to which pure strategy 1 of PL1 is a best response. Then, in the BiClique instance we started with, consider the sets

$$\bar{V} := \Big\{v \in V : \mathbf{r}(v) > \frac{1}{m-K}\frac{1}{K+1}\Big\}\,, \tag{9}$$

and

$$\bar{W} := \Big\{w \in W : \mathbf{s}(w) > \frac{1}{n-K}\frac{1}{K+1}\Big\}\,.$$

Then, we have for all $v \in \bar{V}$ and $w \in \bar{W}$:

$$\mathbf{r}(v)(m-K)(K+1) \cdot \mathbf{s}(w)(n-K)(K+1) > 1\,.$$

Therefore, since $(\mathbf{r}, \mathbf{s})$ satisfies condition (8) by assumption, we get for all $v \in \bar{V}$ and $w \in \bar{W}$ that $(v,w) \in E$. Further below, we show $|\bar{V}|, |\bar{W}| \geq K$. Therefore, choose any $V^* \subseteq \bar{V}$ and $W^* \subseteq \bar{W}$ with $|V^*| = K = |W^*|$, and (V^*, W^*) makes a biclique certificate of the BiClique instance. This shows that the BiClique is therefore a yes instance as well.

Proving the subclaim that $|\bar{V}|, |\bar{W}| \geq K$*:* We only prove $|\bar{V}| \geq K$ since $|\bar{W}| \geq K$ is proven analogously. We derive

$$\begin{aligned}1 &= \sum_{v \in V} \mathbf{r}(v) = \sum_{v \in V^*} \mathbf{r}(v) + \sum_{v \notin V^*} \mathbf{r}(v)\\ &\overset{(6),(9)}{\leq} \sum_{v \in \bar{V}} \frac{1}{K+1} + \sum_{v \notin \bar{V}} \frac{1}{m-K}\frac{1}{K+1}\\ &= |\bar{V}|\frac{1}{K+1} + (m - |\bar{V}|)\frac{1}{m-K}\frac{1}{K+1}\,.\end{aligned}$$

Recall that $(m-K)(K+1) > 0$ since we are not in Cases 1 and 2 (resp. 3). Moving all terms to one side in the above inequality chain and multiplying it by $(m-K)(K+1)$ yields

$$\begin{aligned}0 &\leq (m-K)|\bar{V}| + (m - |\bar{V}|) - (m-K)(K+1)\\ &= m|\bar{V}| - K|\bar{V}| + m - |\bar{V}| - mK - m + K^2 + K\\ &= m(|\bar{V}| - K) - K(|\bar{V}| - K) - (|\bar{V}| - K)\\ &= (m-K-1)(|\bar{V}| - K)\,.\end{aligned}$$

By Cases 1, 2 (resp. 3), and 4 (resp. 5), we have $K \leq m-2$. Therefore, we can divide the above inequality chain by $m-K-1$ to obtain $|\bar{V}| \geq K$. □

Proposition 4.2 allows us to study the following decision problem next.

Definition (CheckIfSameBRs)**.** Given two 3-player normal-form games with strategy set $S^1 \times S^2 \times S^3$, do they have the same best response sets?

Theorem. CheckIfSameBRs *is* **co-NP***-hard.*

Proof. We show that its complement, which we denote as CheckIfDiffBRs, is **NP**-hard by a reduction from CheckIfEverBR. Given two 3-player normal-form games with strategy set $S^1 \times S^2 \times S^3$ and utility functions $(u_i)_i$ and $(u_i')_i$ respectively, CheckIfDiffBRs asks whether there exists a mixed strategy profile $\mathbf{s}^{-i} \in \Delta(S^{-i})$ for some player $i \in \{1,2,3\}$ such that the best response sets $\mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ and $\mathrm{BR}_{u_i'}(\mathbf{s}^{-i})$ differ.

Let G be an instance of CheckIfEverBR, that is, a 3-player normal-form game. Denote its strategy set with $S^1 \times S^2 \times S^3$ and its utility functions with $(u_i)_i$. Determine a strict lower bound $L := -1 + \min_{\mathbf{s} \in \Delta(S)} \big\{u_1(\mathbf{s})\big\}$ on the utilities PL1 may receive in G.

Construct another game G' with the same strategy set as G and with utility functions $u'_2 := u_2$, $u'_3 := u_3$, and

$$u_1(k_1, k_2, k_3) := \begin{cases} L & \text{if } k_1 = 1 \\ u_1(k_1, k_2, k_3) & \text{if } k_1 \neq 1 \end{cases}$$

for all $(k_1, k_2, k_3) \in S$. Then (G, G') shall be the corresponding CheckIfDiffBRs instance. Let us prove equivalence.

Suppose G is a yes instance of CheckIfEverBR. Let $(\mathbf{s}^2, \mathbf{s}^3) \in \Delta(S^2) \times \Delta(S^3)$ be the strategy certificate to which pure strategy 1 of PL1 is a best response, i.e., $1 \in \mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3)$. PL1 has a second strategy (by the definition of a game), and by construction of L, strategy 1 is strictly dominated by strategy 2 in G' for PL1. Therefore, 1 can never be a best response in G' for PL1. In particular, $1 \notin \mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3)$. Hence, (G, G') is a yes instance of CheckIfDiffBRs as well.

Suppose (G, G') is a yes instance of CheckIfDiffBRs. Since PL2 and PL3 receive the same utilities in G and G', their best response sets will be equal. Therefore, the difference in best response sets must be for PL1, that is, there exists a strategy certificate $(\mathbf{s}^2, \mathbf{s}^3) \in \Delta(S^2) \times \Delta(S^3)$ for which $\mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3) \neq \mathrm{BR}_{u'_1}(\mathbf{s}^2, \mathbf{s}^3)$. By Corollary 13, this means that the two sets do not contain the same pure best responses. Let us treat three imaginable situations (which are not mutually exclusive) separately.

Situation 1: We have $1 \in \mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3)$ but $1 \notin \mathrm{BR}_{u'_1}(\mathbf{s}^2, \mathbf{s}^3)$. Then, we are done because this shows that G is also a yes instance of CheckIfEverBR.

Situation 2: There exists a pure strategy $k \in S^1 \setminus \{1\}$ with $k \in \mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3)$ but $k \notin \mathrm{BR}_{u'_1}(\mathbf{s}^2, \mathbf{s}^3)$. Note that u'_1 only differs from u_1 in how much utility strategy $1 \in S^1$ yields under u'_1 and u_1, namely, less under u'_1. Thus, if $k \neq 2$ was a maximizer of $u_1(\cdot, \mathbf{s}^2, \mathbf{s}^3)$, then it must still be a maximizer of $u'_1(\cdot, \mathbf{s}^2, \mathbf{s}^3)$. So this situation will never occur because its premise will never hold.

Situation 3: There exists a pure strategy $k \in S^1$ with $k \in \mathrm{BR}_{u'_1}(\mathbf{s}^2, \mathbf{s}^3)$ but $k \notin \mathrm{BR}_{u_1}(\mathbf{s}^2, \mathbf{s}^3)$. Then $k \neq 1$ because $1 \in S^1$ is never a best response in G'. Moreover, since the only change from u_1 to u'_1 is a decreased payoff of strategy $1 \in S^1$, this Situation 3 can only happen if $1 \in S^1$ is the sole maximizer of $u_1(\cdot, \mathbf{s}^2, \mathbf{s}^3)$. Thus, we are done because this shows that G is also a yes instance of CheckIfEverBR.

In conclusion, we have shown overall that CheckIfDiffBRs is **NP**-hard, and thus, CheckIfSameBRs is **co-NP**-hard. □

B PROOFS OF SECTION 7

To our knowledge, the results and proofs of this section are all novel.

First, some further notation for this appendix. We may write $u \equiv \lambda$ to refer to a function $u : \mathcal{D} \longrightarrow \mathbb{R}$ that is a constant function on its domain $\mathcal{D}$, set to the value $\lambda \in \mathbb{R}$. Moreover, let $e_k \in \mathbb{R}^n$ ($M \in \mathbb{N}$) stand for the k-th standard basis vector, i.e., with a 1 in its k-th entry and 0's anywhere else. Finally, in a game G and for a player i with pure strategy set $S^i = [m_i]$, we identify any pure strategy $k \in [m_i]$ with its corresponding mixed strategy vector in $\Delta(S^i)$ which is exactly the basis vector $e_k \in \mathbb{R}^{m_i}$.

Proposition. *Let $H = \{H^i\}_{i \in [N]}$ be a separable game transformation that universally preserves Nash equilibrium sets and consider the map H^i of a player i. Then H^i only depends on the strategy choices of the opponents. Moreover, H^i universally preserves best responses.*

Proof. Take a separable game transformation $H = \{H^i\}_{i \in [N]}$ that universally preserves Nash equilibrium sets and fix some player i.

1. Fix a pure strategy choice $\mathbf{k}_{-i} \in S^{-i}$ of the opponent players and take some arbitrary value $z \in \mathbb{R}$. Consider the game $G = \{u_j\}_{j \in [N]}$ with constant utility functions $u_j \equiv z$ for all $j \in [N]$. Then, the Nash equilibrium set will be the whole strategy space $\Delta(S)$. By assumption on H, the transformed game $H(G)$ also has the full strategy space as its set of Nash equilibria. In particular, each of the strategy profiles $(1, \mathbf{k}_{-i}), \dots, (m_i, \mathbf{k}_{-i})$ will be a Nash equilibrium of the transformed game $H(G)$. Hence, for all $2 \leq l \leq m_i$:

$$\begin{aligned} h^i_{1,k_{-i}}(z) &\overset{u_i \equiv z}{=} h^i_{1,\mathbf{k}_{-i}}\Big(u_i(1, \mathbf{k}_{-i})\Big) \overset{(2)}{=} H^i(u_i)(1, \mathbf{k}_{-i}) \\ &\overset{\text{Nash-Eq}}{=} \max_{t^i \in \Delta(S^i)} \Big\{ H^i(u_i)(t^i, \mathbf{k}_{-i}) \Big\} \overset{\text{Nash-Eq}}{=} H^i(u_i)(l, \mathbf{k}_{-i}) \\ &= h^i_{l,\mathbf{k}_{-i}}\Big(u_i(l, \mathbf{k}_{-i})\Big) = h^i_{l,\mathbf{k}_{-i}}(z)\,. \end{aligned}$$

Since z and l were chosen arbitrarily, we get

$$h^i_{1,\mathbf{k}_{-i}} = \ldots = h^i_{m_i,\mathbf{k}_{-i}}\,.$$

2. Fix player i's utility function u_i and the opponents' strategy choices $\mathbf{s}^{-i} \in \Delta(S^{-i})$. Then by C.1, it suffices to identify the pure strategies in the best response sets $\mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ and $\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i})$.

Complete the prefixed u_i to a full game $G = \{u_j\}_{j \in [N]}$ by setting $u_j \equiv 0$ for the other players $j \neq i$. Then, the best response set of a player $j \neq i$ is her whole strategy space $\Delta(S^j)$. By assumption on the game transformation H, we get for a pure strategy $e_l = l \in S^i$:

$$\begin{aligned} & e_l \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \\ & \iff (e_l, \mathbf{s}^{-i}) \text{ is a Nash equilibrium for the game } G \\ & \iff (e_l, \mathbf{s}^{-i}) \text{ is a Nash equilibrium for the game } H(G) \\ & \overset{\text{def}}{\iff} e_l \in \mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) \text{ and } \forall j \neq i \ : \\ & \qquad s^j \in \mathrm{BR}_{H^j(u_j)}(s^1, \dots, s^{j-1}, s^{j+1}, \dots, \\ & \qquad\qquad s^{i-1}, e_l, s^{i+1}, \dots, s^N) \\ & \overset{(*)}{\iff} e_l \in \mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i}) \end{aligned}$$

Let us give some further explanation for step $(*)$. Recall the definition for a strategy s^j, $j \neq i$, to be a best response to the opponents' strategy choices $(s^1, \dots, s^{j-1}, s^{j+1}, \dots, s^{i-1}, s^i := e_l, s^{i+1}, \dots s^N)$:

$$s^j \in \operatorname*{argmax}_{t^j \in \Delta(S^j)} \Big\{ \sum_{\mathbf{k} \in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot h^j_{\mathbf{k}}\big(u_j(\mathbf{k})\big) \Big\}\,.$$

We can show that the term in the argmax is constant in t^j. First, note that the maps $h^j_{\mathbf{k}}$ are independent of player j's action, which, in particular, implies $h^j_{\mathbf{k}} = h^j_{1,\mathbf{k}_{-j}}$. Then, rearranging yields

$$\begin{aligned}
&\sum_{\mathbf{k}\in S} s^1_{k_1}\cdot\ldots\cdot s^{i-1}_{k_{i-1}}\cdot t^i_{k_i}\cdot s^{i+1}_{k_{i+1}}\cdot\ldots\cdot s^N_{k_N}\cdot h^j_{\mathbf{k}}\big(u_j(\mathbf{k})\big)\\
&\overset{u_j\equiv 0}{=} \sum_{\mathbf{k}_{-j}}\Bigg(s^1_{k_1}\cdot\ldots\cdot s^{j-1}_{k_{j-1}}\cdot s^{j+1}_{k_{j+1}}\cdot\ldots\cdot s^N_{k_N}\\
&\qquad\qquad\qquad\cdot h^j_{1,\mathbf{k}_{-j}}(0)\cdot\sum_{k_j=1}^{m_j} t^j_{k_j}\Bigg)\\
&\overset{(\dagger)}{=} \sum_{\mathbf{k}_{-j}} s^1_{k_1}\cdot\ldots\cdot s^{j-1}_{k_{j-1}}\cdot s^{j+1}_{k_{j+1}}\cdot\ldots\cdot s^N_{k_N}\cdot h^j_{1,\mathbf{k}_{-j}}(0)\,.
\end{aligned}$$

Since the term in the argmax is constant in t^j, any strategy of player j is a best response to $(s^1,\ldots,s^{j-1},s^{j+1},\ldots,s^{i-1},e_l,s^{i+1},\ldots s^N)$. Therefore, we obtain the equivalence $(*)$ by removing/adding the redundant condition on each s^j, $j\neq i$, to be a best response.

All in all, we proved that the sets $\mathrm{BR}_{u_i}(\mathbf{s}^{-i})$ and $\mathrm{BR}_{H^i(u_i)}(\mathbf{s}^{-i})$ contain the same pure strategies. Corollary C.1 therefore yields set equality. □

Lemma. *Suppose a map H^i universally preserves best responses. Then H^i only depends on the strategy choices of the opponents.*

Proof. Let the pure strategy choice of the opponents be $\mathbf{k}_{-i}\in S^{-i}$. Pick some $z\in\mathbb{R}$ and set $u_i\equiv z$. Then we can reformulate

$$\begin{aligned}
&h^i_{1,\mathbf{k}_{-i}}(z)=\ldots=h^i_{m_i,\mathbf{k}_{-i}}(z)\\
&\iff \forall l\in[m_i]:\ h^i_{l,\mathbf{k}_{-i}}(z)=\max_{p\in[m_i]} h^i_{p,\mathbf{k}_{-i}}(z)\\
&\overset{u_i\equiv z}{\iff} \forall l\in[m_i]:\\
&\qquad h^i_{l,\mathbf{k}_{-i}}(u_i(l,\mathbf{k}_{-i}))=\max_{p\in[m_i]} h^i_{p,\mathbf{k}_{-i}}(u_i(p,\mathbf{k}_{-i}))\\
&\iff \forall l\in[m_i]:\ H^i(u_i)(l,\mathbf{k}_{-i})=\max_{p\in[m_i]} H^i(u_i)(p,\mathbf{k}_{-i})\\
&\iff \forall l\in[m_i]:\ e_l=l\in\mathrm{BR}_{H^i(u_i)}(s^{-i}=\mathbf{k}_{-i})\\
&\overset{(*)}{\iff} \forall l\in[m_i]:\ e_l=l\in\mathrm{BR}_{u_i}(s^{-i}=\mathbf{k}_{-i})\\
&\iff \forall l\in[m_i]:\ u_i(l,\mathbf{k}_{-i})=\max_{p\in[m_i]} u_i(p\mathbf{k}_{-i})\\
&\overset{u_i\equiv z}{\iff} \forall l\in[m_i]:\ z=\max_{p\in[m_i]} z\,.
\end{aligned}$$

In $(*)$, we use that H^i is universally best response preserving.

With the last line of the equivalence chain above being a universally true statement, we obtain that the first line also holds true. Since z was chosen arbitrarily, we can conclude $h^i_{1,k_{-i}}=\ldots=h^i_{m_i,k_{-i}}$. □

Remark. A distance distortion function $\Delta h^1_{\mathbf{k}_{-1}}$, as defined in (4), is skew-symmetric:

$$\forall\, z,w\in\mathbb{R}:\quad \Delta h^1_{\mathbf{k}_{-1}}(z,w)=-\Delta h^1_{\mathbf{k}_{-1}}(w,z)\,. \tag{10}$$

The upcoming lemma reveals an important preliminary observation on how the distance distortion functions $\Delta h^1_{\mathbf{k}_{-1}}$ relate to each other. It highlights how the distorted utility distances are affected by a strategy change of a player $j\neq 1$ from, e.g., some pure strategy $k_j\in[m_j]\setminus\{1\}$ to their pure strategy $1\in[m_j]$.

We formulate the lemma with index variables $\mathbf{p}_{-1}=(p_2,\ldots,p_N)$ instead of $\mathbf{k}_{-1}=(k_2,\ldots,k_N)$ in order to avoid confusion in the proof of the subsequent Lemma 7.5.

Lemma B.1. *Suppose transformation map H^1 universally preserves best responses. Take a player $j\in[N]\setminus\{1\}$ and profile $\mathbf{p}_{-1}\in S^{-1}$ with $p_j\neq 1$. Define $\mathbf{p}'_{-1}\in S^{-1}$ to be the same as $\mathbf{p}_{-1}$ except for player j's choice which shall be set to $p'_j=1$. Then, for all $z,z',w,w'\in\mathbb{R}$:*

$$z-w\geq z'-w'\quad\iff\quad \Delta h^1_{\mathbf{p}_{-1}}(z,w)\geq\Delta h^1_{\mathbf{p}'_{-1}}(z',w')\,.$$

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 7.3, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents. Fix $j,\mathbf{p}_{-1},\mathbf{p}'_{-1}$ and z,z',w,w' as described in the lemma statement. We will construct a utility function u_1 for these parameters such that a universally best response preserving H^1 reveals to satisfies the property of this lemma.

Set $u_1(1,\mathbf{p}_{-1}):=z$ and $u_1(1,\mathbf{p}'_{-1}):=w'$. Additionally, for all pure strategies $l\in[m_1]\setminus\{1\}$, set $u_1(l,\mathbf{p}_{-1}):=w$ and $u_1(l,\mathbf{p}'_{-1}):=z'$. All these utility value assignments are possible because of $p_j\neq 1=p'_j$. The utility payoffs of PL1 (i.e., the values of u_1) from other pure strategy outcomes $\mathbf{k}\in S$ can be set arbitrarily. Finally, consider the opponents' mixed strategy profile $\mathbf{s}^{-1}:=\frac{1}{2}\mathbf{p}_{-1}+\frac{1}{2}\mathbf{p}'_{-1}\in\Delta(S^{-1})$. Then we derive:

$$\begin{aligned}
&z-w\geq z'-w'\\
&\iff \forall l\in[m_1]\setminus\{1\}\ :\\
&\qquad u_1(1,\mathbf{p}_{-1})-u_1(l,\mathbf{p}_{-1})\geq u_1(l,\mathbf{p}'_{-1})-u_1(1,\mathbf{p}'_{-1})
\end{aligned}$$

Reorder and divide by 2

$$\begin{aligned}
&\iff \forall l\in[m_1]\setminus\{1\}\ :\\
&\qquad \frac{1}{2}u_1(1,\mathbf{p}_{-1})+\frac{1}{2}u_1(1,\mathbf{p}'_{-1})\\
&\qquad\qquad \geq\frac{1}{2}u_1(l,\mathbf{p}_{-1})+\frac{1}{2}u_1(l,\mathbf{p}'_{-1})\\
&\iff \forall l\in[m_1]\setminus\{1\}\ :\ u_1(e_1,\mathbf{s}^{-1})\geq u_1(e_l,\mathbf{s}^{-1})\\
&\iff e_1\in\mathrm{BR}_{u_1}(\mathbf{s}^{-1})
\end{aligned}$$

H^1 is universally preserves best responses

$$\begin{aligned}
&\iff e_1\in\mathrm{BR}_{H^1(u_1)}(\mathbf{s}^{-1})\\
&\iff \forall l\in[m_1]\setminus\{1\}\ :\\
&\qquad H^1(u_1)(e_1,\mathbf{s}^{-1})\geq H^1(u_1)(e_l,\mathbf{s}^{-1})\\
&\iff \forall l\in[m_1]\setminus\{1\}\ :\\
&\qquad \frac{1}{2}h^1_{1,\mathbf{p}_{-1}}(u_1(1,\mathbf{p}_{-1}))+\frac{1}{2}h^1_{1,\mathbf{p}'_{-1}}(u_1(1,\mathbf{p}'_{-1}))\\
&\qquad\qquad \geq\frac{1}{2}h^1_{l,\mathbf{p}_{-1}}(u_1(l,\mathbf{p}_{-1}))+\frac{1}{2}h^1_{l,\mathbf{p}'_{-1}}(u_1(l,\mathbf{p}'_{-1}))\\
&\iff \forall l\in[m_1]\setminus\{1\}\ :\\
&\qquad h^1_{1,\mathbf{p}_{-1}}(z)+h^1_{1,\mathbf{p}'_{-1}}(w')\geq h^1_{l,\mathbf{p}_{-1}}(w)+h^1_{l,\mathbf{p}'_{-1}}(z')
\end{aligned}$$

H^1 does not depend on the pure strategy choice of player 1

$$\iff h^1_{\mathbf{p}_{-1}}(z) + h^1_{\mathbf{p}'_{-1}}(w') \geq h^1_{\mathbf{p}_{-1}}(w) + h^1_{\mathbf{p}'_{-1}}(z')$$
$$\iff h^1_{\mathbf{p}_{-1}}(z) - h^1_{\mathbf{p}_{-1}}(w) \geq h^1_{\mathbf{p}'_{-1}}(z') - h^1_{\mathbf{p}'_{-1}}(w')$$
$$\iff \Delta h^1_{\mathbf{p}_{-1}}(z, w) \geq \Delta h^1_{\mathbf{p}'_{-1}}(z', w')$$

□

Lemma. *Suppose transformation H^1 universally preserves best responses. Then the pure-strategy maps in H^1 equally distort distances:*

$$\forall \mathbf{k}_{-1} \in S^{-1} \,:\, \Delta h^1_{\mathbf{k}_{-1}} = \Delta h^1_{\mathbf{1}_{-1}}$$

where $\mathbf{1}_{-1} := (1, \dots, 1) \in S^{-1}$.

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 7.3, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents. Fix $\mathbf{k}_{-1} \in S^{-1}$. Recall that the elements $j \geq 2$ and $\mathbf{p} \in S^{-1}$ in B.1 can be chosen arbitrarily[5]. So we can apply B.1 on a trivially true statement to get for all $z, w \in \mathbb{R}$:

$$\begin{aligned}
& z - w \geq z - w \\
& \implies \forall j \in [N] \setminus \{1\} : \\
& \qquad \Delta h^1_{k_2,\dots,k_{j-1},k_j,1,\dots,1}(z, w) \\
& \qquad\quad \geq \Delta h^1_{k_2,\dots,k_{j-1},1,1,\dots,1}(z, w) \\
& \implies \Delta h^1_{k_2,\dots,k_{N-1},k_N}(z, w) \geq \Delta h^1_{k_2,\dots,k_{N-1},1}(z, w) \\
& \qquad\quad \geq \dots \geq \Delta h^1_{1,\dots,1}(z, w) \,.
\end{aligned}$$

With skew-symmetry, we similarly obtain

$$\begin{aligned}
& w - z \geq w - z \\
& \implies \forall j \in [N] \setminus \{1\} : \\
& \qquad \Delta h^1_{k_2,\dots,k_{j-1},k_j,1,\dots,1}(w, z) \\
& \qquad\quad \geq \Delta h^1_{k_2,\dots,k_{j-1},1,1,\dots,1}(w, z) \\
& \implies \Delta h^1_{k_2,\dots,k_{N-1},k_N}(w, z) \geq \Delta h^1_{k_2,\dots,k_{N-1},1}(w, z) \\
& \qquad\quad \geq \dots \geq \Delta h^1_{1,\dots,1}(w, z) \\
& \overset{\cdot(-1)}{\implies} \Delta h^1_{k_2,\dots,k_{N-1},k_N}(z, w) \leq \Delta h^1_{1,\dots,1}(z, w) \,.
\end{aligned}$$

Putting both together, we have for all $z, w \in \mathbb{R}$:

$$\begin{aligned}
\Delta h^1_{\mathbf{k}_{-1}}(z, w) &= \Delta h^1_{k_2,\dots,k_{N-1},k_N}(z, w) \\
&= \Delta h^1_{1,\dots,1}(z, w) = \Delta h^1_{\mathbf{1}_{-1}}(z, w) \,.
\end{aligned}$$

□

Lemma. *Suppose transformation H^1 universally preserves best responses. Then we obtain for all $\mathbf{k}_{-1} \in S^{-1}$ that*

(1) map $h^1_{\mathbf{k}_{-1}}$ is strictly increasing, and that
(2) map $h^1_{\mathbf{k}_{-1}}$ distorts distances independently of their reference points:

$$\forall z, z', \lambda \in \mathbb{R} \,:\, \Delta h^1_{\mathbf{k}_{-1}}(z + \lambda, z) = \Delta h^1_{\mathbf{k}_{-1}}(z' + \lambda, z') \,. \tag{11}$$

[5] We required $p_j \neq 1$, but this is irrelevant for the argument we are making here.

Proof. Take a transformation map H^1 that universally preserves the best response sets. Then by Lemma 7.3, its maps $h^1_{\mathbf{k}}$ only depend on the strategy choices $\mathbf{k}_{-1}$ of the opponents.

1. Let us first consider $h^1_{2,1,\dots,1}$ that is associated to the pure strategy profile $(2, 1, \dots, 1) \in S^{-1}$. Apply B.1 in the upcoming line $(*)$ with parameters $j = 2$, $\mathbf{p}_{-1} = (2, 1, \dots, 1)$, and $z' = w' \in \mathbb{R}$ to get for arbitrary $z, w \in \mathbb{R}$:

$$\begin{aligned}
z \geq w &\iff z - w \geq 0 = z' - w' \\
&\overset{(*)}{\iff} \Delta h^1_{2,1,\dots,1}(z, w) \geq \Delta h^1_{\mathbf{1}_{-1}}(z', w') \overset{z'=w'}{=} 0 \\
&\iff h^1_{2,1,\dots,1}(z) \geq h^1_{2,1,\dots,1}(w) \,.
\end{aligned}$$

Consequently, we have for arbitrary $\bar{z}, \bar{w} \in \mathbb{R}$:

$$\begin{aligned}
\bar{z} > \bar{w} &\iff \bar{z} \geq \bar{w} \text{ and } \bar{w} \ngeq \bar{z} \\
&\overset{\text{by above}}{\iff} h^1_{2,1,\dots,1}(\bar{z}) \geq h^1_{2,1,\dots,1}(\bar{w}) \\
&\qquad \text{and } h^1_{2,1,\dots,1}(\bar{w}) \ngeq h^1_{2,1,\dots,1}(\bar{z}) \\
&\iff h^1_{2,1,\dots,1}(\bar{z}) > h^1_{2,1,\dots,1}(\bar{w}) \,.
\end{aligned}$$

This shows that $h^1_{2,1,\dots,1}$ is strictly increasing.
For arbitrary $\mathbf{k}_{-1} \in S^{-1}$, we can then use Lemma 7.5 to obtain

$$\begin{aligned}
\bar{z} > \bar{w} &\iff h^1_{2,1,\dots,1}(\bar{z}) > h^1_{2,1,\dots,1}(\bar{w}) \\
&\iff \Delta h^1_{2,1,\dots,1}(\bar{z}, \bar{w}) > 0 \\
&\iff \Delta h^1_{\mathbf{k}_{-1}}(\bar{z}, \bar{w}) = \Delta h^1_{\mathbf{1}_{-1}}(\bar{z}, \bar{w}) \\
&\qquad = \Delta h^1_{2,1,\dots,1}(\bar{z}, \bar{w}) > 0 \\
&\iff h^1_{\mathbf{k}_{-1}}(\bar{z}) > h^1_{\mathbf{k}_{-1}}(\bar{w}) \,.
\end{aligned}$$

Thus, $h^1_{\mathbf{k}_{-1}}$ is strictly increasing as well.

2. Because of Lemma 7.5, we only need to show that the map $\Delta h^1_{\mathbf{1}_{-1}}$ satisfies property (11), which would consequently imply the property for all maps $\Delta h^1_{\mathbf{k}_{-1}}$.

Fix $z, z', \lambda \in \mathbb{R}$. Then the upcoming equivalence chain uses skew-symmetry (10) in $(*)$, Lemma 7.5 in $(\dagger)$, and B.1 in $(\star)$ for parameters $j = 2$ and $\mathbf{p}_{-1} = (2, 1, \dots, 1)$:

$$\begin{aligned}
& \Delta h^1_{\mathbf{1}_{-1}}(z + \lambda, z) = \Delta h^1_{\mathbf{1}_{-1}}(z' + \lambda, z') \\
& \overset{(*)}{\iff} \Delta h^1_{\mathbf{1}_{-1}}(z + \lambda, z) \geq \Delta h^1_{\mathbf{1}_{-1}}(z' + \lambda, z') \\
& \qquad \text{and } \Delta h^1_{\mathbf{1}_{-1}}(z, z + \lambda) \geq \Delta h^1_{\mathbf{1}_{-1}}(z', z' + \lambda) \\
& \overset{(\dagger)}{\iff} \Delta h^1_{2,\dots,1}(z + \lambda, z) \geq \Delta h^1_{\mathbf{1}_{-1}}(z' + \lambda, z') \\
& \qquad \text{and } \Delta h^1_{2,\dots,1}(z, z + \lambda) \geq \Delta h^1_{\mathbf{1}_{-1}}(z', z' + \lambda) \\
& \overset{(\star)}{\iff} z + \lambda - z \geq z' + \lambda - z' \\
& \qquad \text{and } z - (z + \lambda) \geq z' - (z' + \lambda) \,.
\end{aligned}$$

The last line is a true statement and thus, the first line as well. Because $z, z', \lambda \in \mathbb{R}$ were taken arbitrarily, map $h^1_{\mathbf{1}_{-1}}$ satisfies property (11). □

Theorem. *Let $H = \{H^i\}_{i\in[N]}$ be a separable game transformation. Then:*

$$\begin{aligned} &H \text{ universally preserves Nash equilibrium sets} && \text{(i)}\\ \iff &\text{for each player } i\text{, map } H^i \text{ universally} && \text{(ii)}\\ &\text{preserves best responses}\\ \iff &H \text{ is a positive affine transformation.} && \text{(iii)} \end{aligned}$$

Proof. See main body. □

C HELPING LEMMAS

This appendix section does not contain original ideas and is just included for completeness.

Denote the restriction of a best response set to its pure strategies as $\mathrm{PBR}_{u_i}(\mathbf{s}^{-i}) := \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \cap \{e_1, \ldots, e_{m_i}\}$. Then, we have that best responses are always convex combinations of pure best responses:

Lemma. *Take a game $G = \{u_i\}_{i\in[N]}$, fix a player $i \in [N]$ and a strategy profile $\mathbf{s}^{-i} \in \Delta(S^{-i})$ of the opponents. Then, we have for $t^i \in \Delta(S^i)$:*

$$\begin{aligned} &t^i \in \mathrm{BR}_{u_i}(\mathbf{s}^{-i})\\ &\iff \forall k \in [m_i] : \; t^i_k = 0 \;\text{ or }\; e_k \in \mathrm{PBR}_{u_i}(\mathbf{s}^{-i})\,. \end{aligned} \tag{12}$$

Proof. We can observe

$$\begin{aligned} &u_i(t^i, \mathbf{s}^{-i})\\ &= \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k})\\ &= \sum_{k_i=1}^{m_i} t^i_{k_i} \cdot \sum_{\mathbf{k}_{-i}\in S^{-i}} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots\\ &\qquad\qquad \cdot s^N_{k_N} \cdot u_i(\mathbf{k})\\ &= \sum_{k_i=1}^{m_i} t^i_{k_i} \cdot u_i(e_{k_i}, \mathbf{s}^{-i}) = \sum_{k=1}^{m_i} t^i_k \cdot u_i(e_k, \mathbf{s}^{-i})\,. \end{aligned} \tag{13}$$

Thus, the mixed strategy t^i of player i only determines the convex combination of the attainable utility values $\left(u_i(e_k, \mathbf{s}^{-i})\right)_k$. Therefore, any best response strategy t^i must only randomize over maximal values within $\left(u_i(e_k, \mathbf{s}^{-i})\right)_k$, that is, over pure best response strategies. □

Corollary C.1. *Two best response sets (of possibly different games) are equal if and only if they contain the same pure best responses.*

Lemma. *Take a PAT $H_{\mathrm{PAT}} = \left\{\alpha^i, C^i\right\}_{i\in[N]}$ and any game $G = \{u_i\}_{i\in[N]}$. Then, the transformed game $H_{\mathrm{PAT}}(G) = \{u'_i\}_{i\in[N]}$ has the same best response sets as G. Consequently, $H_{\mathrm{PAT}}(G)$ also has the same Nash equilibrium set as G.*

Proof. The proof is an appropriate generalization of the known proof for Lemma 5.1.

Take a game $\{u_i\}_{i\in[N]}$, fix a player i and the opponents' strategy choices $\mathbf{s}^{-i}$. Then, we have

$$\begin{aligned} &\mathrm{BR}_{u'_i}(\mathbf{s}^{-i}) = \operatorname*{argmax}_{t^i\in\Delta(S^i)} \left\{ u'_i(t^i, \mathbf{s}^{-i}) \right\}\\ &= \operatorname*{argmax}_{t^i\in\Delta(S^i)} \Big\{\\ &\qquad \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u'_i(\mathbf{k}) \Big\}\\ &\overset{(1)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)} \Big\{ \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots\\ &\qquad\qquad \cdot s^N_{k_N} \cdot \left(\alpha^i \cdot u_i(\mathbf{k}) + c^i_{\mathbf{k}_{-i}}\right) \Big\}\\ &\overset{(*)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)} \Big\{\\ &\qquad \alpha^i \cdot \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k})\\ &\qquad + \sum_{\mathbf{k}_{-i}\in S^{-i}} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot c^i_{\mathbf{k}_{-i}} \cdot 1 \Big\}\\ &\overset{(\dagger)}{=} \operatorname*{argmax}_{t^i\in\Delta(S^i)} \Big\{\\ &\qquad \sum_{\mathbf{k}\in S} s^1_{k_1} \cdot \ldots \cdot s^{i-1}_{k_{i-1}} \cdot t^i_{k_i} \cdot s^{i+1}_{k_{i+1}} \cdot \ldots \cdot s^N_{k_N} \cdot u_i(\mathbf{k}) \Big\}\\ &= \operatorname*{argmax}_{t^i\in\Delta(S^i)} \left\{ u_i(t^i, \mathbf{s}^{-i}) \right\} = \mathrm{BR}_{u_i}(\mathbf{s}^{-i}) \end{aligned}$$

We obtain the second summand in $(*)$ by changing the order of summation and multiplication such that $\sum_{k_i=1}^{m_i} t_i$ remains as the most inner sum. Since $\sum_{k_i=1}^{m_i} t_i = 1$, this factor can be dropped. We get line $(\dagger)$ because the argmax operator is neither affected by a constant in t_i (such as the secoond summand) nor by rescaling with a positive factor (such as α_i).

Finally, the definition of a Nash equilibrium immediately implies that strategy profile s is a Nash equilibrium for the PAT transformed game $\{u'_i\}_{i\in[N]}$ if and only if it was one for the original game $\{u_i\}_{i\in[N]}$. □

D MONOTONE AND ADDITIVE IMPLIES LINEAR

The proof of the following lemma is taken from ProofWiki [34, 35] and just included for completeness.

Lemma. *Take a map $h : \mathbb{R} \longrightarrow \mathbb{R}$ which is monotone and additive. Then:*

(1) $h(0) = 0$.
(2) $\forall x \in \mathbb{R} : \quad -h(-x) = h(x)$.
(3) $\forall n \in \mathbb{N}, x \in \mathbb{R} : \quad h(n \cdot x) = n \cdot h(x)$.
(4) $\forall p \in \mathbb{Z}, x \in \mathbb{R} : \quad h(p \cdot x) = p \cdot h(x)$.
(5) $\forall r \in \mathbb{Q}, x \in \mathbb{R} : \quad h(r \cdot x) = r \cdot h(x)$.
(6) $\forall x \in \mathbb{R} : \quad h(x) = x \cdot h(1)$.

In particular, the last conclusion yields that h is linear.

Proof. The first three conclusions follow from h being additive.

1.

$$h(0) = h(0) + h(x) - h(x) = h(0 + x) - h(x) = 0\,.$$

2.

$$\begin{aligned}\forall x \in \mathbb{R}\ :\quad -h(-x) &= -\Big(h(-x) + h(x)\Big) + h(x)\\ &= -h(-x + x) + h(x)\\ &= -h(0) + h(x)\\ &= h(x)\,.\end{aligned}$$

3. Proof by induction. The induction start $n = 1$ is clear, so assume it to be true for $n \in \mathbb{N}$.
Then, for all $x \in \mathbb{R}$:

$$\begin{aligned}h\Big((n+1)\cdot x\Big) &= h(n \cdot x + x) = h(n \cdot x) + h(x)\\ &= n \cdot h(x) + h(x) = (n+1)\cdot h(x)\,.\end{aligned}$$

4. The statement for the case $p \in \mathbb{Z} \cap \{z \geq 0\}$ follows from the first and third conclusion. If $p \in \mathbb{Z} \cap \{z < 0\}$, we can use the second and third conclusion to obtain for all $x \in \mathbb{R}$:

$$\begin{aligned}h(p \cdot x) &= h\Big((-p)\cdot(-x)\Big) = (-p)\cdot h(-x)\\ &= (-p)\cdot\Big(-h(x)\Big) = p \cdot h(x)\,.\end{aligned}$$

5. Write $r = \frac{p}{q}$ where $p \in \mathbb{Z}, q \in \mathbb{N}$. Then, by the fourth conclusion:

$$\begin{aligned}h(r \cdot x) &= \frac{1}{q}\cdot q \cdot h\Big(\frac{p}{q}\cdot x\Big) = \frac{1}{q} h\Big(q \cdot \frac{p}{q}\cdot x\Big) = \frac{1}{q} h(p \cdot x)\\ &= \frac{1}{q}\cdot p \cdot h(x) = r \cdot h(x)\,.\end{aligned}$$

6. Suppose $x \in \mathbb{Q}$. Then, the fifth conclusion yields

$$h(x) = h(x \cdot 1) = x \cdot h(1)\,.$$

Therefore, suppose $x \in \mathbb{R} \setminus \mathbb{Q}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can take an increasing sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ that converges to x (from below) and a decreasing sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ that converges to x (from above). In the case where h is an increasing function, we have for all $n \in \mathbb{N}$:

$$\begin{aligned}r_n \leq x \leq s_n &\implies h(r_n) \leq h(x) \leq h(s_n)\\ &\implies r_n \cdot h(1) \leq h(x) \leq s_n \cdot h(1)\,.\end{aligned}$$

Taking the limit $n \to \infty$ in the last inequality chain yields

$$x \cdot h(1) \leq h(x) \leq x \cdot h(1)\,.$$

If h is a decreasing function instead of an increasing one, we get the same implications but with reverse inequalities in the second and last inequality chains. The end result, however, will be the same. Putting everything together yields the sixth conclusion.

□

Corollary. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be monotone and satisfy for all $z, z', \lambda \in \mathbb{R}$:*

$$h(z + \lambda) - h(z) = h(z' + \lambda) - h(z')\,. \tag{14}$$

Then h is affine linear, i.e., there exist some $a, c \in \mathbb{R}$ such that for all $z \in \mathbb{R}\ :\ h(z) = az + c$.

Proof. Define $h'(z) := h(z) - h(0)$, which is still a monotone function. By our assumption on h, we have for all $x, y \in \mathbb{R}$:

$$\begin{aligned}h'(x + y) &= h(x + y) - h(0)\\ &= h(x + y) - h(y) + h(y) - h(0)\\ &= h(x) - h(0) + h(y) - h(0)\\ &= h'(x) + h'(y)\,.\end{aligned}$$

Therefore, we can apply Lemma 7.7 to h' to get $a \in \mathbb{R}$ such that for all $z \in \mathbb{R}$

$$\begin{aligned}h(z) &= h(z) - h(0) + h(0) = h'(z) + h(0)\\ &= az + h(0) =: az + c\,.\end{aligned}$$

□

E UTILITY THEORY IN GAME THEORY

This section revises some related utility theory and is just included for completeness. A proper treatment can be found in e.g. Mas-Colell et al. [25].

Preferences and Utility Functions. Suppose a decision maker can choose one outcome from a space C of N-many outcomes (where N finite). Moreover, the decision maker prefers some outcomes over others which is captured by her preference relation $\succeq$ on C.

We typically describe the preferences of the decision maker through utility functions:

Definition E.1. A utility function $u : C \longrightarrow \mathbb{R}$ is said to represent a preference relation $\succeq$ if for all $c, d \in C$, we have $c \succeq d \iff u(c) \geq u(d)$.

Multiple utility functions can represent the same preference relation. Their practical use is that they translate the preference relation $\succeq$ into comparisons of numerical values.

On the other hand, starting with a utility function u yields an induced preference relation $\succeq$ through

$$\forall\, c, d \in C \quad : \quad c \succeq d :\iff u(c) \geq u(d)\,.$$

Lotteries and the Expected Utility. Now suppose we want to allow the decision maker to choose each outcome in C with some probability. Call such a probability distribution $L = (p_1, \dots, p_N)$ over C a lottery. The i-th outcome in C can then be represented by the lottery $e_i \in \mathbb{R}^n$. Thus, we extended the choice space of the decision maker from C to the space $\mathcal{L}$ of lotteries. We can also extend Definition E.1 to preference relations $\succeq$ over $\mathcal{L}$ by requiring $u : \mathcal{L} \longrightarrow \mathbb{R}$ and $\forall\, L, M \in \mathcal{L}\ :\ L \succeq M \iff u(L) \geq u(M)$.

We will be especially interested in those utility functions that simply compute the expected utility of randomly choosing an outcome according to L.

Definition. A *von Neumann-Morgenstern (NM)* expected utility function is a map $U : \mathcal{L} \longrightarrow \mathbb{R}$ that is determined by its values $U(e_i)$ on the outcomes $e_i \in C, i \in [N]$, and by

$$\forall L = (p_1, \dots, p_N)\ :\quad U(L) = \sum_{i=1}^{N} p_i \cdot U(e_i)\,.$$

The following theorem describes the preference relations that can be represented by a NM expected utility function. The theorem relies on four properties - called *axioms* - that a preference relation $\succeq$ can satisfy: Completeness[6], Transitivity[7], Continuity[8] and Independence[9].

Theorem 3 (Expected Utility Theorem)**.** *Let preference relation $\succeq$ satisfy the four axioms mentioned above. Then $\succeq$ can be represented by a NM expected utility function U. Moreover, the representing U is unique up to a positive affine transformation. That is, if U and U' are NM expected utility functions representing $\succeq$, then there exist $\alpha, c \in \mathbb{R}$ such that for all $L \in \mathcal{L}$, we have $U'(L) = \alpha \cdot U(L) + c$.*

Proof. See Proposition 6.B.2 and 6.B.3 from Mas-Colell et al. [25]. □

In contrast to Theorem 3, suppose we start with an arbitrary NM expected utility function U. Then U induces a preference relation $\succeq$ on $\mathcal{L}$ by

$$\forall L, L' \in \mathcal{L} \quad : \quad L \succeq L' :\Longleftrightarrow U(L) \geq U(L') \, .$$

By construction, U represents $\succeq$. One can also show that this induced preference relation $\succeq$ satisfies the four axioms. Therefore, by Theorem 3, U uniquely represents the induced $\succeq$ up to a PAT.

Connections to Game Theory. Take a multiplayer game $G = \big(N, \{S^i\}_{i \in [N]}, \{u_i\}_{i \in [N]}\big)$. Then, the utility functions u_i induce each player's preferences according to the following paragraphs:

Consider a game that only allows for pure strategy play. Then, given some player i and the pure strategy profile s^{-i} of the opponents, the "sliced" utility function $u_i(\cdot, s^{-i})$ induces a preference relation $\succeq$ for player i over her strategy set S^i.

Now suppose that we allow for mixed strategy play in the games. In that case, each element in $\Delta(S^i)$ can be viewed as a lottery over the choice set $C := S^i$. Moreover, player i's utility payoff from a mixed strategy profile $s \in \bigtimes_{i=1}^{N} \Delta(S^i)$ is

$$u_i(s^i, s^{-i}) = \sum_{k_i=1}^{m_i} s^i_{k_i} \cdot u_i(e_{k_i}, s^{-i}) \, .$$

Therefore, $u_i(\cdot, s^{-i})$ has the form of a NM expected utility function. This induces a preference relation $\succeq_{i,s^{-i}}$ on the space of lotteries $\Delta(S^i)$ with $\succeq_{i,s^{-i}}$ satisfying the four axioms. Hence, $u_i(\cdot, s^{-i})$ represents the induced preference relation $\succeq_{i,s^{-i}}$ uniquely up to a PAT.

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[6]For all $L, M \in \mathcal{L}$, we have $L \succeq M$ or $L \preceq M$ (or both, in which case we write $L \sim M$).

[7]For all $L, M, N \in \mathcal{L}$, if $L \succeq M$ and $M \succeq N$, then $L \succeq N$.

[8]For all $L, M, N \in \mathcal{L}$ with $L \succeq M \succeq N$, there exists probability $p \in [0, 1]$ such that $p \cdot L + (1 - p) \cdot N \sim M$.

[9]For all $L, M, N \in \mathcal{L}$ and $p \in [0, 1]$, we have $L \succeq M$ if and only if $p \cdot L + (1 - p) \cdot N \succeq p \cdot M + (1 - p) \cdot N$.